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Solving the Combinatorial Explosion Problem When Calculating the Multiple-Hit Vulnerability of Aircraft

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Two precise calculation methods (Markov chain or tree diagram) are commonly used for assessing aircraft vulnerability to multiple hits by nonexplosive penetrators. Because the dimension of Markov transition matrix or the number of tree branches increases exponentially along with the increasing number of redundant components, when the total number of redundant components reaches a certain amount, the "combinatorial explosion" is unavoidable. This paper, using the Monte Carlo technique, proposes a method for solving the combinatorial explosion problem. This method simulates all of the existing states of aircraft to Model of Filling Boxes with Balls; by randomly and uniformly sampling the threat hit locations, the aircraft cumulated probability of kill can be attained. Two provided examples show the correctness and feasibility of the proposed method. Analysis shows that the developed method with high accuracy apparently costs a short amount of time compared with the precise methods, when the total number of the redundant components is relatively large. Moreover, the CPU run time increases almost linearly along with the increasing number of redundant components, and the combinatorial explosion is avoided, and so this method is more applicable to engineering computation.

I. Introduction

AIRCRAFT combat survivability (ACS)[1,2] is defined as the capability of an aircraft to avoid or withstand a man-made hostile environment. Survivability is composed of two focus areas, which are susceptibility and vulnerability. One of the important parts in aircraft vulnerability assessment is to calculate the cumulated probability of kill at the threat of multiple hits by nonexplosive penetrators (multiple-hit vulnerability). At present, there are two precise methods, namely, the Markov chain method, which is also called states transition matrix method, and the tree diagram method, for calculating the multiple-hit vulnerability.[1–6] Markov chain and tree diagram calculate the cumulated probability of kill of aircraft progressively according to the sequence of the threat hits.[1] When the total number of redundant components of aircraft is small, the speed for calculating the multiple-hit vulnerability using the two precise methods is very fast. However, the "combinatorial explosion" will happen, and time cost becomes unacceptably long when the total number of redundant components reaches a certain amount, because the dimension of Markov transition matrix or the exist states of aircraft, and the number of tree branches increases exponentially along with the increasing number of redundant components.

Monte Carlo (MC) simulation is a powerful tool for dealing with the stochastic problems. In this paper, utilizing the Monte Carlo method and simulating all of the exist states of aircraft to "Model of Filling Boxes with Balls,"[7] by randomly and uniformly sampling the threat hit locations, the cumulated probability of kill of an aircraft can be attained, and the combinatorial explosion problem is overcome when calculating multiple-hit vulnerability.

II. Analysis of the Aircraft Exist States

In the precise calculation methods for aircraft multiple-hit vulnerability, the determination of the number of tree branches in tree diagram or the dimension of the states transition matrix in Markov chain is based on the exist states of aircraft when hit by threats. References 1 and 2 have given a typical illustration of determining the dimension of the states transition matrix. An aircraft example consists of four critical components: pilot, fuel tank, engine 1, and engine 2, of which pilot and fuel tank are nonredundant, and engines 1 and 2 are the only set of mutually redundant components. The example aircraft will exist in the following five states: 1) nonredundant critical components (the pilot and the fuel tank) have been killed, 2) only engine 1 has been killed, 3) only engine 2 has been killed, 4) both engines 1 and 2 have been killed, and 5) none of the critical components are killed. In our opinion, because both state 1 and state 3 result in kill of aircraft, the two states can be merged into one state, namely, aircraft kill state. Hence, actually the example aircraft will exist in the four unique states: 1) aircraft has been killed, 2) only engine 1 has been killed, 3) only engine 2 has been killed, and 4) none of the critical components have been killed.

The preceding example reveals that the number of aircraft exist states is related to the exist state of each critical component. Because kill of any nonredundant critical component or any set of redundant critical components will result in kill of the aircraft, the kill event of any nonredundant critical component or any set of redundant critical components can be merged into one event, namely, aircraft kill event. Hence, the number of the exist states totally depends on the total number of redundant components and is related to the number of combinations of exist states of redundant componnents. In the following illustration, the aircraft states are referred to the merged and unique exist states.

III. Assumptions

Aircraft multiple-hit vulnerability is based on the single-hit vulnerability.[1,2] The single-hit vulnerability of aircraft for a particular threat aspect is usually expressed as the probability the aircraft is killed given a random (uniformly distributed) hit anywhere on the presented area A_P of the aircraft $P_{K/H}$, or as the vulnerable area A_V. A_V is related to $P_{K/H}$ by[1]

$$A_V = A_P P_{K/H} \tag{1}$$

where A_P denotes the projected area of aircraft in the plane normal to the approach direction of threat. The computation of $P_{K/H}$ and A_V is based on the vulnerable area corresponding to each exist state of aircraft given a single hit. This paper is based on the assumptions that 1) the vulnerable area corresponding to each exist state and the projected area of aircraft given a single hit are known and the detailed calculation methods can be found in Refs. 1, 2, and 8, and 2) the

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threats of the aircraft we considered are nonexplosive penetrators, such as projectiles or fragments of missile, etc.

IV. Multiple-Hit Vulnerability Calculating Method Based on Monte Carlo Technique

To solve the combinatorial explosion problem, we propose the Monte Carlo based calculation method. The flowchart of the proposed method is shown in Fig. 1, and more details about the flowchart are presented in the following.

A. Constructing the Analytical Model and Reference Coordinate System

Assume the aircraft exists in $n+2$ states. The first state refers to the state that any nonredundant component or any set of redundant components is killed, and for ease of illustration we call it "kill state." The second state refers to the state that none of the critical components is killed, and we call it "nonkill state." The last n states refer to the states of the redundant components and the combinatorial states among them (except the kill state and the nonkill state), and we call the n states "redundant states." Figure 2 shows the "Model of Filling Boxes with Balls" for aircraft multiple-hit vulnerability calculation. The upper part of the figure is corresponding to nonkill state, the middle part of the figure is corresponding to redundant states, and the lower part of the figure is corresponding to kill state.

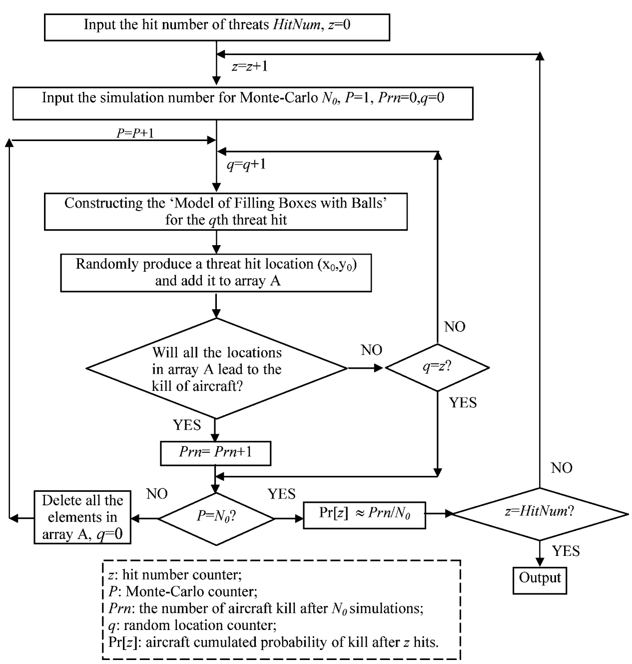

Fig. 1 Monte Carlo based method for calculating aircraft cumulated probability of kill.

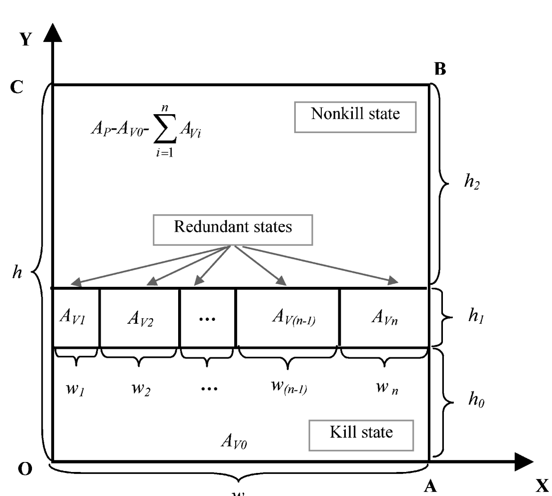

Fig. 2 Model of filling boxes with balls[7] for aircraft multiple-hit vulnerability calculation.

The model of "Filling Boxes with Balls" is originally designed for deducing the aircraft's equivalent singly vulnerable area.[7] In this model, the area of each box is equal to the component's vulnerable area. By randomly and uniformly filling boxes with balls, the expected number of hits required to kill an aircraft can be given, and the aircraft equivalent singly vulnerable area can be attained. In this paper, the Monte Carlo based method simulates all of the exist states of aircraft to Model of Filling Boxes with Balls, and by randomly and uniformly producing the hit locations the multiple hit vulnerability can be obtained.

Let h and w be the height and the width of the model, respectively; h_0 be the height of the box corresponding to the vulnerable area of kill state; h_1 be the height of the boxes corresponding to the vulnerable areas of redundant states; h_2 be the height of the box corresponding to the area of nonkill state; A_{Vi} and W_i be the vulnerable area and the associated width of the box corresponding to the ith redundant state, respectively; A_{V0} be vulnerable area of the kill state; and

$$A_P - A_{V0} - \sum_{i=1}^{n} A_{Vi}$$

be the area of the nonkill state.

As can be seen from Fig. 2, the sum of the areas of the three kinds of states is the aircraft presented area A_p. Let

$$w = h = \sqrt{Ap} \tag{2}$$

Then,

$$h_0 = \frac{A_{V0}}{w} \tag{3}$$

$$h_1 = \frac{\left(\sum_{i=1}^{n} A_{Vi}\right)}{w} \tag{4}$$

$$h_2 = h - h_0 - h_1 \tag{5}$$

$$w_i = \frac{A_{Vi}}{h_1} \qquad (i = 1, 2, 3, \ldots, n) \tag{6}$$

To produce the random hit locations, which are two dimensionally and uniformly distributed in rectangle OABC of Fig. 2, for Monte Carlo simulation we should build the reference coordinate system XOY. The selection of coordinate origin and the coordinate axis depends on the ease of calculation, and we choose the coordinate system shown in Fig. 2 for illustration of our proposed method.

B. Criterion for the Kill of Component

As is shown in Fig. 1, array A has recorded all of the random threat hit locations in each MC simulation. According to the Model of Filling Boxes with Balls,[7] when the box contains the random hit location (x_0, y_0), the exist state corresponding to the box will happen, which implies the components corresponding to that state will be killed. Hence, once knowing the state happens or not, we can determine whether the component(s) is (are) killed or not. The criterion for the occurrence of the kill state is the location (x_0, y_0) satisfies

$$0 < x_0 < w \tag{7}$$

and

$$0 < y_0 < h_0 \tag{8}$$

The criterion for the occurrence of the ith redundant state, whose vulnerable area is A_{Vi}, is the location (x_0, y_0) satisfies

$$\sum_{r=0}^{i-1} w_r < x_0 < \sum_{t=0}^{i} w_t \tag{9}$$

and

$$h_0 < y_0 < h_0 + h_1 \tag{10}$$

where

$$w_0 = 0 \tag{11}$$

Table 1 Vulnerable area for each exist state (m^2)

Exist states	Aircraft 1	Aircraft 2 ($d1, d2, d3, d4$)	Aircraft 3 ($b3, b4$)	Aircraft 4 ($b1, b2$)	Aircraft 5 ($d6, d7$)	Aircraft 6 ($d8, d9$)	Aircraft 7 ($d10, d11$)	Aircraft 8 ($d12, d13$)	Aircraft 9 ($d14, d15$)
A_{V0}	8.606343	8.381343	6.562905	5.935084	5.866334	5.804834	5.762084	5.719626	5.695876
$d1$	——	0.010000	0.010000	0.010000	0.010000	0.010000	0.010000	0.010000	0.010000
$d2$	——	0.052500	0.052500	0.052500	0.052500	0.052500	0.052500	0.052500	0.052500
$d3$	——	0.030000	0.030000	0.030000	0.030000	0.030000	0.030000	0.030000	0.030000
$d4$	——	0.132500	0.132500	0.132500	0.132500	0.132500	0.132500	0.132500	0.132500
$b3$	——	——	0.906001	0.906001	0.906001	0.906001	0.906001	0.906001	0.906001
$b4$	——	——	0.912437	0.912437	0.912437	0.912437	0.912437	0.912437	0.912437
$b1$	——	——	——	0.313427	0.313427	0.313427	0.313427	0.313427	0.313427
$b2$	——	——	——	0.314394	0.314394	0.314394	0.314394	0.314394	0.314394
$d6$	——	——	——		0.035000	0.035000	0.035000	0.035000	0.035000
$d7$	——	——	——		0.033750	0.033750	0.033750	0.033750	0.033750
$d8$	——	——	——	——	——	0.019050	0.019050	0.019050	0.019050
$d9$	——	——	——	——	——	0.030785	0.030785	0.030785	0.030785
$b4, d9$	——	——	——	——	——	0.011215	0.011215	0.011215	0.011215
$b3, d8$	——	——	——	——	——	0.000450	0.000450	0.000450	0.000450
$d10$	——	——	——	——	——	——	0.015604	0.015604	0.015604
$d11$	——	——	——	——	——	——	0.018150	0.018150	0.018150
$b4, d11$	——	——	——	——	——	——	0.003600	0.003600	0.003600
$b3, d10$	——	——	——	——	——	——	0.005396	0.005396	0.005396
$d12$	——	——	——	——	——	——	——	0.031734	0.031734
$d13$	——	——	——	——	——	——	——	0.007507	0.007507
$b4, d13$	——	——	——	——	——	——	——	0.003217	0.003217
$d14$	——	——	——	——	——	——	——	——	0.022500
$d15$	——	——	——	——	——	——	——	——	0.001250
nonkill	73.701157	73.701157	73.701157	73.701157	73.701157	73.701157	73.701157	73.701157	73.701157

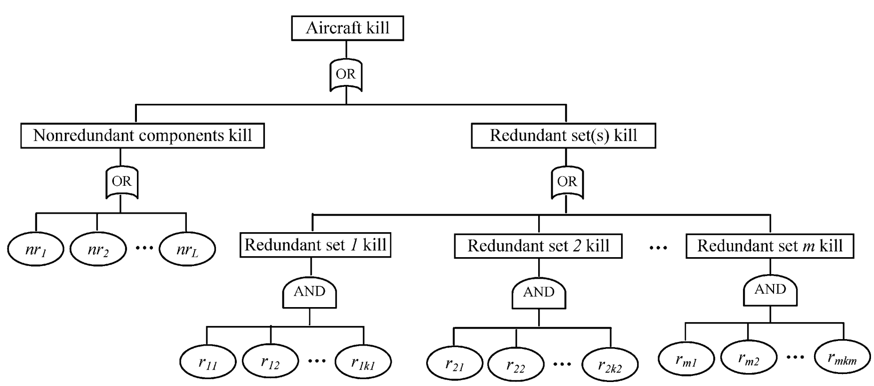

Fig. 3 Typical aircraft kill tree.

C. Criterion for the Kill of Aircraft

After determination of the exist state of each vulnerable component, the exist state of the aircraft can be determined by the kill tree.[1] Kill tree is a visual illustration of the critical components and all component redundancies. The kill tree illustrates the number of redundant components in a redundant set that must be killed to cause an aircraft kill. For example, when an aircraft consists of L nonredundant components ($nr_1, nr_2, \ldots, nr_L$) and m sets of redundant components, assuming the ith redundant set has ki components and the kill of all the components in each set can lead to kill of aircraft, the kill tree can be constructed using the Boolean operators such as AND and OR, as is shown in Fig. 3, where r_{ij} denotes the jth component in the ith redundant set.

D. Calculating the Vulnerability Using Monte Carlo Simulations

As is shown in Fig. 1, Prn denotes the number of the kill of aircraft for the hit number z after N_0 MC simulations. When the simulation number N_0 is sufficient enough, the aircraft cumulated probability of kill after z hits $Pr[z]$ can be approximately determined by

$$Pr[z] \approx Prn/N_0 \tag{12}$$

V. Examples

This section contains two examples. The first is a testing example for verifying the correctness of the proposed method by the comparisons of MC computation results and the precise results obtained by Markov chain method. The second example is used to illustrate the feasibility and practicability of the proposed method by the comparisons of the CPU run-time cost (on Pentium 4.0/1.6-Hz personal computer) by the proposed method and that cost by the precise Markov chain method on the same computer.

A. Original Data

For ease of comparisons, we assume there are nine aircraft (aircraft 1, aircraft 2, ..., and aircraft 9) with a different number of redundant components. The presented area of each aircraft for the given threat hit aspect is 82.307500 m^2, and the kill tree for each aircraft can be described as shown in Fig. 3. Aircraft 1 has zero redundant components. Aircraft 2 has one redundant set including four mutually redundant components ($d1, d2, d3$, and $d4$). Aircraft 3 has two redundant sets. The first set is the same as the one of aircraft 2, and the second set contains two mutually redundant components ($b3$ and $b4$). The following aircraft contain an increasing set of two mutually redundant components based on the former aircraft. The vulnerable area corresponding to each exist state for each aircraft is shown in Table 1. In Table 1, the codes in the parentheses of the first row refer to the increased components of the aircraft in the column based on the former aircraft. The codes listed in the first column refer to the different exist states. A_{V0} denotes the aircraft kill state and *nonkill* denotes the nonkill state. Other codes denote

Table 2 Comparisons of computation results using the Monte Carlo method and the precise method

Aircraft designator	Hit number	Computation results: Precise	MC $N_0 = 10,000$	MC $N_0 = 50,000$	MC $N_0 = 100,000$	Relative errors, %: $N_0 = 10,000$	$N_0 = 50,000$	$N_0 = 100,000$
1 (0 redundant set)	1	0.104563	0.111000	0.101680	0.104500	6.156101	2.757184	0.060246
	2	0.198193	0.195500	0.197220	0.197930	1.358775	0.490936	0.132702
	3	0.282033	0.280500	0.281920	0.283220	0.543554	0.040070	0.420872
	4	0.357106	0.353100	0.358040	0.357840	1.121795	0.261548	0.205542
	5	0.424329	0.429100	0.428160	0.421370	1.124362	0.902837	0.697339
	10	0.668603	0.667800	0.666620	0.668920	0.120100	0.296587	0.047409
	15	0.809224	0.808000	0.807400	0.806200	0.151254	0.225404	0.373689
	20	0.890176	0.889100	0.886900	0.891290	0.120873	0.368016	0.125145
2 (1 redundant set)	1	0.101830	0.102500	0.101700	0.102070	0.657960	0.127662	0.235685
	2	0.193290	0.193700	0.193260	0.195180	0.212119	0.015519	0.977807
	3	0.275437	0.275000	0.274100	0.277050	0.158654	0.485407	0.585612
	4	0.349219	0.351400	0.345600	0.346370	0.624535	1.036308	0.815816
	5	0.415488	0.411200	0.412400	0.414050	1.032044	0.743222	0.346097
	10	0.658345	0.654100	0.656980	0.659010	0.644796	0.207339	0.101012
	15	0.800299	0.801800	0.800400	0.800180	0.187558	0.012624	0.014866
	20	0.883272	0.883000	0.884000	0.882850	0.030792	0.082422	0.047777
4 (3 redundant sets)	1	0.072109	0.069900	0.072020	0.072090	3.063418	0.123420	0.026348
	2	0.139291	0.143700	0.139300	0.139140	3.165316	0.006462	0.108412
	3	0.201854	0.200400	0.203560	0.201700	0.720328	0.845160	0.076294
	4	0.260089	0.260200	0.259200	0.259570	0.042671	0.341807	0.199550
	5	0.314273	0.315700	0.313880	0.315780	0.454062	0.125052	0.479524
	10	0.533058	0.534400	0.535280	0.532660	0.251755	0.416840	0.074660
	15	0.683949	0.683600	0.685820	0.681640	0.051025	0.273557	0.337593
	20	0.787198	0.783000	0.787220	0.788280	0.533286	0.002794	0.137450
8 (7 redundant sets)	1	0.069491	0.067700	0.070780	0.069330	2.577313	1.854920	0.231684
	2	0.134433	0.130900	0.135360	0.135240	2.628079	0.689564	0.600301
	3	0.195095	0.195300	0.194800	0.195550	0.105075	0.151208	0.233216
	4	0.251733	0.245300	0.252920	0.250990	2.555489	0.471530	0.295155
	5	0.304590	0.303700	0.302100	0.302670	0.292192	0.817487	0.630351
	10	0.519843	0.515000	0.520500	0.520530	0.931627	0.126389	0.132156
	15	0.670497	0.671600	0.671680	0.672150	0.164502	0.176432	0.246536
	20	0.775084	0.776200	0.774880	0.774370	0.143982	0.026323	0.092120

Table 3 Time cost comparisons of Monte Carlo method and precise method

Aircraft designator	Total number of redundant components	Time cost, s: MC($N_0 = 10,000$)	MC($N_0 = 50,000$)	MC($N_0 = 100,000$)	Precise	Dimension of Markov transition matrix
1	0	1	2	5	0	2
2	4	1	5	9	0	16
3	6	1	6	11	0	46
4	8	1	8	15	2	136
5	10	1	8	17	11	406
6	12	2	10	21	106	1,216
7	14	3	12	24	976	3,646
8	16	3	13	28	8,763	10,936
9	18	3	14	29	78,888	32,806

the redundant states. In those codes, a single component code denotes the state that only the component has been killed, for example, $d1$ denotes the state that only component $d1$ has been killed; and the combinatorial code denotes the state only the components that combined have been killed, for example, $b3$, $d8$ denotes the state that only components $b3$ and $b8$ have been killed. Along with the increasing number of redundant components, the exist states of the aircraft will increase exponentially. For saving space, Table 1 only lists the states corresponding to which the vulnerable areas are not equal to 0.0.

B. Example 1

The cumulated probabilities of kill after 1–20 threat hits for the four aircraft (aircraft 1, aircraft 2, aircraft 4, and aircraft 8) in Table 1 using the proposed MC simulation method and the precise method are partially listed in Table 2.

From the table, one can see the maximum relative error of the MC simulation result to precise result is 6.156101% for $N_0 = 10{,}000$; 2.757184% for $N_0 = 50{,}000$; and 0.977807% for $N_0 = 100{,}000$. The accuracy of the proposed approximate method is obvious.

C. Example 2

The preceding example has demonstrated the accuracy of the proposed method. The computation time comparisons of the proposed method and the precise method for the nine aircraft listed in Table 1 are shown in Table 3. Table 3 shows that, when the number of redundant components is less than 10, the speed of either of the two methods is fast, and time cost by the precise method is comparable to the time cost by the MC simulations ($N_0 = 100{,}000$). However, when the number of redundant components is more than 12, the time cost by MC simulations is obviously less than the time cost by the precise method. The time cost by the proposed method ($N_0 = 100{,}000$) is 1/5 of the time cost by the precise method when the number of the redundant components is 12, 1/310 when the number of redundant components is 16, and 1/2720 when the number of redundant components is 18.

Figure 4 shows the relationship between the dimension of the Markov matrix and the number of redundant components. Figure 5 shows the relationship between the time cost and the number of redundant components using the precise method. The two figures show that the dimension of Markov transition matrix and the time

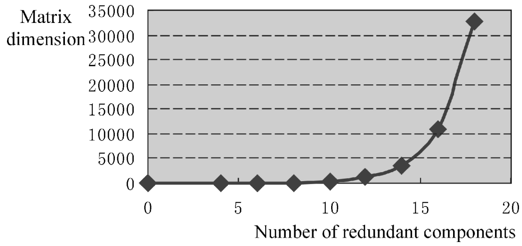

Fig. 4 Relationship between matrix dimension and number of redundant components.

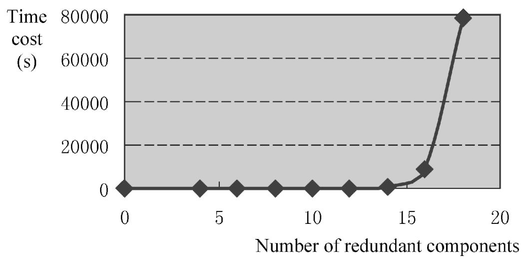

Fig. 5 Relationship between time cost and number of redundant components using precise method.

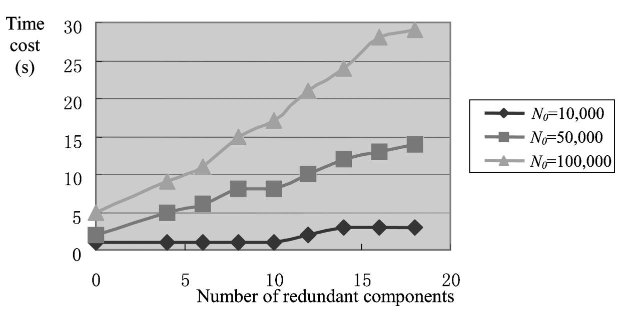

Fig. 6 Relationship between time cost and number of redundant components using the Monte Carlo method.

cost increase exponentially along with the increasing number of redundant components, and this could result in the combinatorial explosion problem.

Figure 6 shows the relationship between the time cost and the number of redundant components using the proposed method. The figure shows that the time cost increases almost linearly along with the increasing number of redundant components, the slope of each curve stays unchanged on the whole, no combinatorial explosion happens, and the computation speed is very fast compared with that obtained by the precise method.

VI. Conclusions

1) The proposed method in this paper, using Monte Carlo simulation technique, has solved the combinatorial explosion problem when calculating aircraft multiple-hit vulnerability.

2) The proposed method can provide high computation accuracy and short computation time cost. The computation time increase almost linearly along with the increasing number of redundant components, so that the combinatorial explosion is avoided, and thus this method is more applicable to engineering computation.

3) The number of aircraft exist states totally depends on the number of all of the redundant components and is related to the number of combinations of exist states of redundant componnents. The precise calculation methods, namely, Markov chain and tree diagram, only apply to the case where the number of redundant components is small. However, along with the development of the computer memory and the CPU speed, the proposed method and the precise methods can be combined together considering the specific computer configuration. Namely, when the number of redundant components is less than a certain amount, about 10–11 for the present examples, the precise methods are the accepted option; otherwise, the MC simulation method is recommended to calculate the aircraft multiple-hit vulnerability.

Acknowledgments

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